Analyzing a Spring-Damper System in Earthquake Dynamics

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Introduction

The Quest for Earthquake-Resistant Structures

- **Motivation:** Inspired by the devastation of the 2011 Japan earthquake, exploring ways to minimize structural damage through mathematical models.
- **Objective:** To model the damping system of a building during an earthquake using second-order differential equations, focusing on the effects of various variables on system dynamics.
- **Approach:** Employing the Laplace transform to simplify the computational complexity of the mathematical models, enabling more efficient analysis and simulation.

Mathematical Background

Foundations for Earthquake Damping Systems

- **Second-Order Differential Equations:** Form the basis for modeling physical systems, including earthquake-induced building oscillations.

- **Roots of Auxiliary Equations:** Determine the nature of the system's response, influencing its stability and oscillatory behavior.

- **Laplace Transform:** Facilitates solving differential equations by converting them into algebraic equations, simplifying analysis.

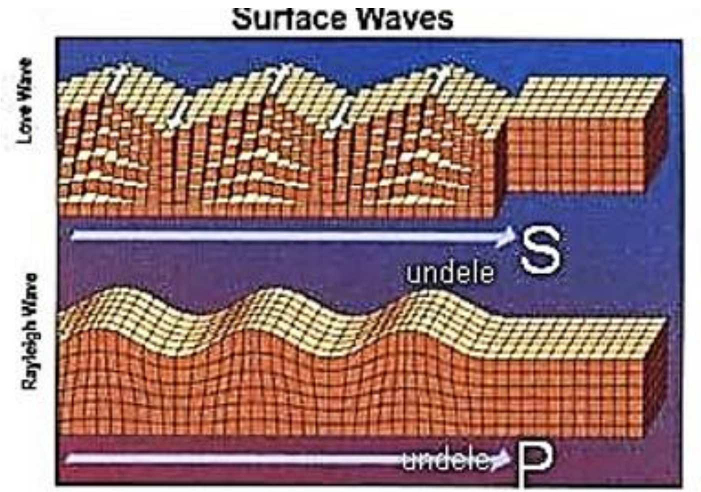
Modeling the Damping System

Spring-Damper Analogy

- **Spring-Damper System:** Analogous to a building's structural damping system, combining elastic (spring) and damping (damper) elements to model seismic response.
- **Force Equilibrium:** Incorporates restoring force (spring), damping force (damper), and external force (earthquake) to establish the system's dynamic equilibrium.
- **System Representation:** Mathematically formulated as a second-order differential equation, capturing the interplay of mass, stiffness, and damping in response to seismic excitation.

Types of Seismic waves

- Earthquakes generate seismic waves that transmit energy.
- Seismic waves are categorized based on the medium they traverse into body waves and surface waves.
- Body waves travel through Earth's interior, comprising:
  - P waves (longitudinal or compressional waves), vibrating parallel to the wave's direction.
  - S waves (transverse waves), vibrating perpendicular to the wave's direction.
- Surface waves propagate along Earth's surface, slower than body waves, with more vigorous particle movement.
- Surface waves include:
  - Love waves (transverse waves), known for causing significant damage, such as building collapses.
  - Rayleigh waves.
- In the proposed model, Love waves are considered to provide sinusoidal horizontal forces on buildings.



APPROACH TO SOLVE THE 2ND ORDER DIFFERENTIAL EQUATION

The differential equation (hereafter referred as DE) in mathematics is an equation involving the derivatives of one variable with respect to the other variable. Specifically, second order differential equation is when an equation is in the form of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$.

The simple case is when the equation is in the form of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

To solve such DE, I will deduce a suitable approach using the first order DE.
Suppose there is an equation

$$a\frac{dy}{dx} + by = 0$$

This could be rearranged into

$$a\frac{dy}{dx} =- by$$

Using the method of separating the variables, the above DE can be solved as follows.

$$\frac{1}{y}\frac{dy}{dx} =- \frac{b}{a}$$

Integrating both sides with respect to *x*,

$$\int \frac{1}{y} dy = \int - \frac{b}{a} dx$$

Now this can be solved using integration

$$ln(y) = -\frac{b}{a}x + C$$

Assuming that C is equal to $ln(A)$

$$ln(y) = -\frac{b}{a}x + ln(A)$$

By putting both sides as index of e

$$e^{ln(y)} = e^{-\frac{b}{a}x + ln(A)}$$

$$y = Ae^{-\frac{b}{a}x}$$

It can be stated that the solution to $a\frac{dy}{dx} + by = 0$ takes the form of $y = Ae^{mx}$

Going back to the second order DE of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, the solution would be similar to the one I got above. Thus, an educated guess of the solution would be $y = Ae^{mx}$

First differentiate the estimated solution by using chain rule

$$y = Ae^{mx}$$

$$\frac{dy}{dx} = Ame^{mx} \quad \frac{d^2y}{dx^2} = Am^2e^{mx}$$

Then substitute these derivatives and y into $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

$$Am^2ae^{mx} + Ambe^{mx} + Ace^{mx} = 0$$

$$Ae^x(am^2 + bm + c) = 0$$

Since Ae^x is greater than zero, $am^2 + bm + c$ needs to be zero.

$$am^2 + bm + c = 0$$

This quadratic equation, known as the auxiliary equation, can be used to solve the DE.

Implications of the Roots of Auxiliary equation(Complementary Function)

Depending on the discriminant of the auxiliary equation, the solution to 2nd order DE of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, known as complementary functions, can take three different forms.

1. If $b^2 - 4ac > 0$, the auxiliary equation has two distinct real roots, α and β.

The existence of two distinct real roots for the auxiliary equation means that the solution takes the form of $y = Ae^{\alpha x}$ and $y = Be^{\beta x}$. Since $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$, a general solution to DE can be achieved by adding the two solutions.

Therefore, the general solution for $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is

$$y = Ae^{\alpha x} + Be^{\beta x}$$

where A and B are unknown constants.

2. If $b^2 - 4ac = 0$, there are two repeated roots, α

The solution would be $y = Ae^{\alpha x}$. However, there is another solution $y = Bxe^{\alpha x}$ that satisfies the DE. Therefore, the general solution becomes

$$y = (A + Bx)e^{ax}$$

3. If $b^2 - 4ac < 0$, there are two complex roots of the form $p \pm iq$

The solutions would be

$$y = Pe^{(p+iq)x} \text{ and } y = Qe^{(p-iq)x}$$

This could be written as follows.

$$y = Pe^{p}(e^{iqx}) \text{ and } y = Qe^{p}(e^{-iqx})$$

Given that Euler form of a complex number is $e^{ix} = cos(x) + isin(x)$ and $e^{-ix} = cos(-x) + isin(-x) = cos(x) - isin(x)$, I can rewrite the solutions as

$$y = Pe^{p}(cos(qx) + isin(qx)) \text{ and } y = Qe^{p}(cos(qx) - isin(qx))$$

Then the general solution becomes

$$y = Pe^{p}(cos(qx) + isin(qx)) + Qe^{p}(cos(qx) - isin(qx))$$

$$y = e^{p}((P + Q)cos(qx) + (P - Q)isin(qx))$$

Let $P + Q = A$ and $(P + Q)i = B$ to make it simple

$$y = e^{p}((A)cos(qx) + (B)sin(qx))$$

Solving 2nd Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

The differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ can be solved through 3 steps.

1. Solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ for complementary functions as explained in the previous section.

2. Then find the “particular integral”, which takes different form depending on $f(x)$.

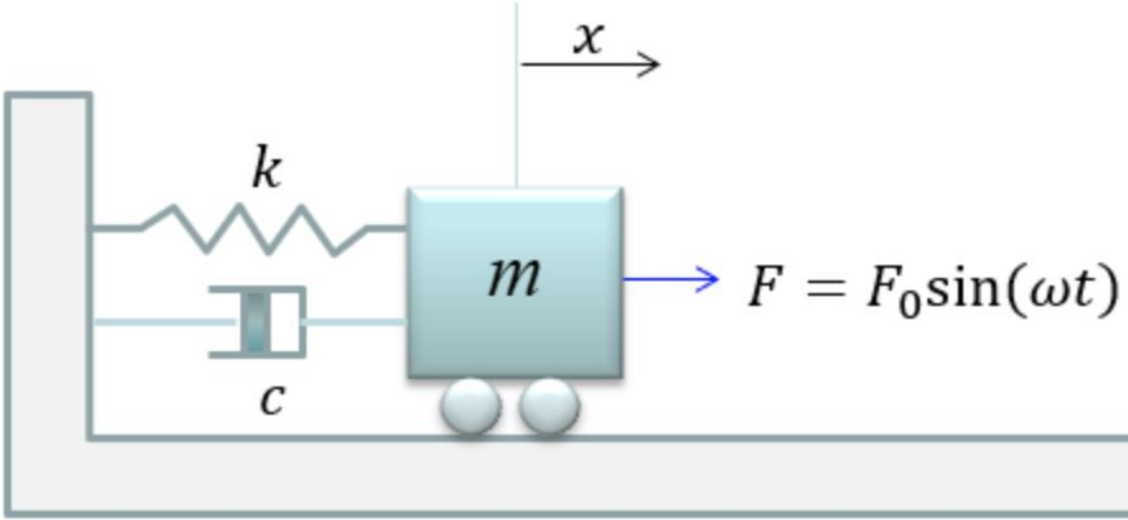
For example, for $f(x) = Fsin(\omega x)$, the particular integral should be sinusoidal and the would take the form of $mcos(\omega x) + nsin(\omega x)$, where m and n are unknown constants. Using this form, corresponding $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ can be found. Then the derivatives and y can be substituted into the DE and by comparing the coefficients, m and n can be evaluated.

3. Finally the general solution to the DE would be

$$y = complementary\ function + particular\ integral$$

MODELLING

MODEL SET UP



There are mainly three forces acting on the building: restoring force, resistive force, and external force due to the earthquake.

1. Hooke's law states that the restoring force is proportional to the extension, x, from the original point(Robert). The direction of the restoring force is such that it opposes the deformation, in this case, extension. So, the f restoring force formula would be $F =- kx$, where k is a spring coefficient and x is the extension.
2. The Resistive force is the force against the movement. In my model, the resistive force causes damping and reduces the oscillation of the mass. The resistive force can be written as $F =- cv$ where c is the viscous damping coefficient and v is velocity. Since velocity is the rate of change in distance, $\frac{dx}{dt}$, the formula can also be written as $F =- c\frac{dx}{dt}$.
3. The external force due to the earthquake is assumed to be the force applied by the Love wave. Since the Love wave is transverse wave on the surface, I am going to assume that the external force is sinusoidal and has the form $Fsin(\omega t)$.

By using the three main forces, the net force acting on the building can be found.

$$F_{resultant} = -c\frac{dx}{dt} - kx + Fsin(\omega t)$$

Newton's second law states that the rate of change in momentum of an object is directly proportional to the resultant force applied to an object and if mass does not change, its formula is $F_{resultant} = ma$(Crowell).

$$ma = -c\frac{dx}{dt} - kx + Fsin(\omega t)$$

Since acceleration is derivative of velocity and velocity is derivative of distance, $F = ma$ can also be written as $F = m\frac{d^2x}{dt^2}$. Now replace this.

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx + Fsin(\omega t)$$

By rearranging this equation, a second order DE is set up.

$$Fsin(\omega t) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

Solution to the Model

Using the method explained in the background information, a solution to this model can be found.

First, for $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$, let $x = Ae^{\alpha t}$ and find its successive derivatives.

$$\frac{dx}{dt} = \alpha Ae^{\alpha t} \qquad \frac{d^2x}{dt^2} = \alpha^2 Ae^{\alpha t}$$

By substituting the following derivatives to the equation, $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$,

$$m\alpha^2 Ae^{\alpha t} + c\alpha Ae^{\alpha t} + kAe^{\alpha t} = 0$$

This can be used to find the auxiliary equation as follows.

$$m\alpha^2 + c\alpha + k = 0$$

Now using the quadratic formula, the solutions to the auxiliary equation can be found.

$$\alpha = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Note that when a building is forced to oscillate due to the sinusoidal force, its displacement would be also sinusoidal. Therefore, for the complementary function to be sinusoidal, **the** α **must be complex.** Therefore, it can be written as,

$$\alpha = \frac{-c}{2m} \pm \frac{\sqrt{c^2-4mk}}{2m}$$

$$\alpha = \frac{-c}{2m} \pm i\frac{\sqrt{4mk-c^2}}{2m}, \ (i = \sqrt{-1})$$

The complementary function becomes

$$y = e^{-\frac{c}{2m}t}\left(A \cos cos \left(\frac{\sqrt{4mk-c^2}}{2m}t\right) + B \sin sin \left(\frac{\sqrt{4mk-c^2}}{2m}t\right)\right) \quad (eq1)$$

For the particular integral, $x = \lambda cos(\omega t) + \mu sin(\omega t)$ (***eq*2**) should be tried, because $f(t) = Fsin(\omega t)$.

$$\frac{dx}{dt} =- \omega\lambda sin(\omega t) + \mu\omega cos(\omega t) \quad \frac{d^2x}{dt^2} =- \omega^2\lambda cos(\omega t) - \mu\omega^2 sin(\omega t)$$

Then substituting them to $Fsin(\omega x) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$

$$m\left(- \omega^2\lambda \cos cos\,(\omega t) - \mu\omega^2 \sin sin\,(\omega t)\right) + c(- \omega\lambda \sin sin\,(\omega t) + \mu\omega cos(\omega t)) + k(\lambda cos(\omega t) + \mu sin(\omega t)) = Fsin(\omega t)$$

By rearranging them,

$$(-\omega^2 m\lambda + c\mu\omega + k\lambda)cos(\omega t) + (-m\mu\omega^2 - c\omega\lambda + k\mu)sin(\omega t) = Fsin(\omega t)$$

$$(-\omega^2 m\lambda + c\mu\omega + k\lambda)cos(\omega t) = 0 \qquad (-m\mu\omega^2 - c\omega\lambda + k\mu)sin(\omega t) = Fsin(\omega t)$$

$$\left(k - \omega^2 m\right)\lambda + c\mu\omega = 0 \quad (eq3) \qquad \left(k - m\omega^2\right)\mu - (c\omega)\lambda = F \quad (eq4)$$

By solving the above two simultaneous equations, the values of μ and λ can be calculated.

ANALYSIS

Variables

The general solution to my model involves many different variables such as mass m, spring constant k, damping coefficient c, and force exerted by the seismic wave F. However, it was almost impossible to find the suitable numerical values for my variables. Hence, instead of finding the particular solution to specific values of m, k, c and F, we are going to vary each variable while keeping the rest constant as 1 in order to see the effect of each variable on the final displacement of the building.

From further research, I have found that the typical Love wave has a frequency of 10 Hz. So ω can be found by using the equation $\omega = 2\pi f$.

$$\omega = 2\pi f = 2\pi \times 10 = 20\pi$$

Base solution where all the variables equal to 1

When m, k, c and F are 1, the complementary function (***eq*1**) becomes

$$x = e^{-\frac{1}{2}t}\left(A \cos cos\left(\frac{\sqrt{3}}{2}t\right) + B \sin sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

To find the particular integral, the two simultaneous equations ***eq*3** and ***eq*4** have to be solved to get λ and μ. Using m, k, c and F as 1 and ω as 20π, the equations become

$$\left(1 - (20\pi)^2\right)\lambda + (20\pi)\mu = 0 \qquad (eq5)$$

$$\left(1 - (20\pi)^2\right)\mu - (20\pi)\lambda = 1 \qquad (eq6)$$

By multiplying (20π) on ***eq*5** and multiplying $(1 - (20\pi)^2)$ on ***eq*6**, two equations can be added to eliminate λ.

$$((20\pi)^2 + (1 - (20\pi)^2)^2)\mu = (1 - (20\pi)^2)$$

$$\mu = \frac{(1-(20\pi)^2)}{(20\pi)^2+(1-(20\pi)^2)^2} = -\ 0.000253$$

Using the μ obtained,

$$(1 - (20\pi)^2)\lambda + (20\pi)\times - 0.000253 = 0$$

$$\lambda =- 0.000004$$

The particular integral, ***eq2***, becomes

$$x =- 0.000004cos(20\pi t) - 0.000253sin(20\pi t)$$

and the general solution is

$$x = e^{-\frac{1}{2}t}\left(A \cos cos\left(\frac{\sqrt{3}}{2}t\right) + B \sin sin\left(\frac{\sqrt{3}}{2}t\right)\right) - 0.000004cos(20\pi t) - 0.000253sin(20\pi t)$$

Now the constant A can be found by using the fact that the displacement of the building is zero when the time is zero.

Substitute $t = 0,\ x = 0$

$$0 = 1\times((A)\times1 + B\times0) - 0.000004\times1 - 0.000253\times0$$

$$A = 0.000004\ (=- \lambda)$$

Now B can be found by using the fact that when the time is $t = \frac{1}{10}$, displacement should be zero as the applied frequency is 10Hz meaning that it completes 1 oscillation in every $\frac{1}{10}$ seconds.

$$0 = e^{-\frac{1}{20}}\left(0.000004 \cos cos\left(\frac{\sqrt{3}}{20}\right) + B \sin sin\left(\frac{\sqrt{3}}{20}\right)\right) - 0.000004cos(2\pi) - 0.000253sin(2\pi)$$

Solving above equation gives $B = 0.000003$

Therefore, the reference solution to my model is

$$x = e^{-\frac{1}{2}t}\left(0.000004 \cos cos\left(\frac{\sqrt{3}}{2}t\right) + 0.000003 \sin sin\left(\frac{\sqrt{3}}{2}t\right)\right) - 0.000004cos(20\pi t) - 0.000253sin(20\pi t)$$

Variation of Mass

We are going to increase the mass of the building to 10 kg and 100 kg. Notice that when changing the variables, the value of constant λ, μ, A and B change. Therefore, further calculations are needed to find the following constants and a new particular solution. The workings for such calculations are lengthy and I have put all my workings in the appendix.

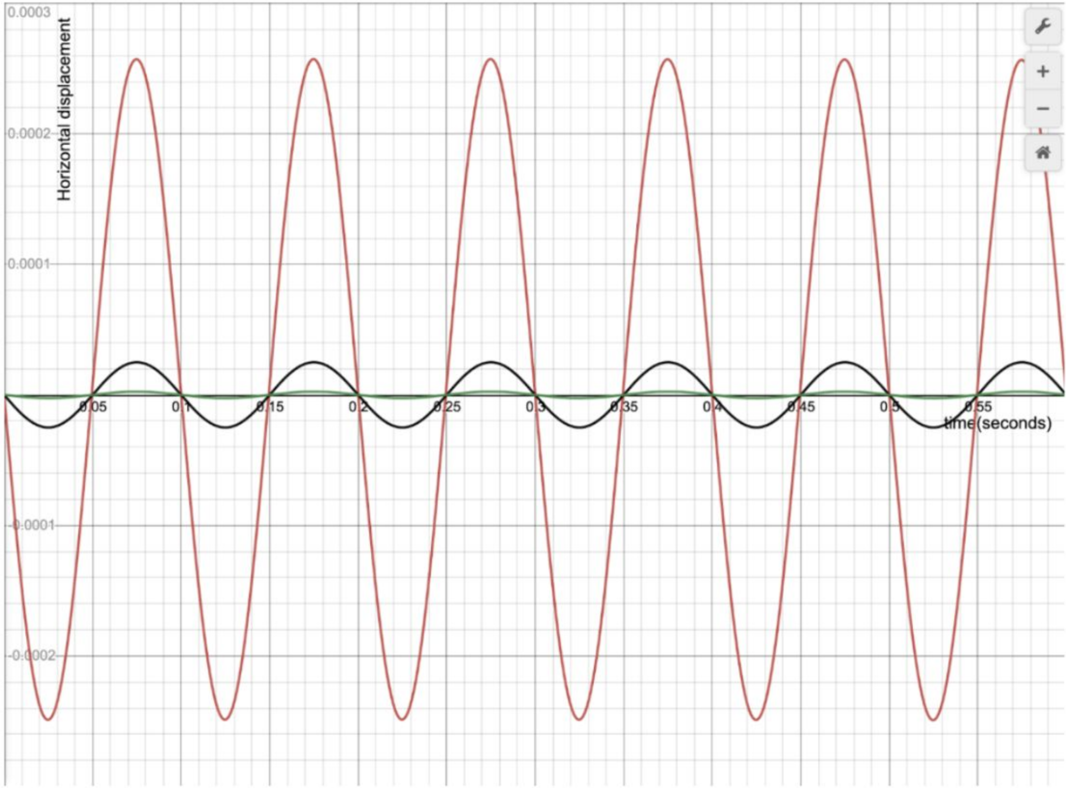
When mass=10kg and other variables are all 1, the particular solution becomes,

$$x = e^{-\frac{1}{20}t}\left((4.04\times10^{-8}) \cos cos\left(\frac{\sqrt{39}}{2}t\right) + (7.12\times10^{-9}) \sin sin\left(\frac{\sqrt{39}}{2}t\right)\right) - (4.04\times10^{-8})cos(20\pi t)$$
$$- (0.000025)sin(20\pi t)$$

Now, when mass=100 kg and other variables are all 1, the particular solution is

$$x = e^{-\frac{1}{200}t}\left((4.03\times10^{-10}) \cos cos\left(\frac{\sqrt{399}}{200}t\right) + (2.22\times10^{-11}) \sin sin\left(\frac{\sqrt{399}}{200}t\right)\right) - (4.03\times10^{-10})cos(20\pi t)$$

$$- (0.0000253)sin(20\pi t)$$



Red: 1 kg Black: 10 kg Green: 100 kg

It can be seen that when the mass of a building increases, the horizontal displacement decreases.

Variation of spring constant

Now, we are going to vary the spring constant while keeping the remaining variables as 1.
Here, we are going to increase the spring constant from 1 to 10 and 100N/m.

Now, when the spring constant, $k = 10$N/m, the particular solution is,

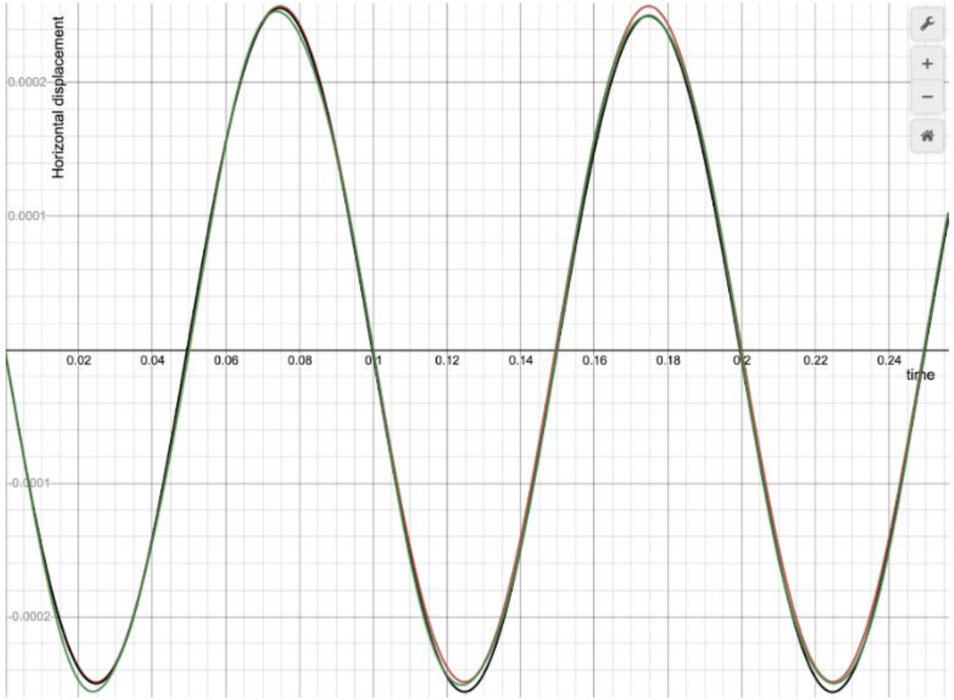
$$x = e^{-\frac{1}{2}t}\left((0.00000405)\cos cos\left(\frac{\sqrt{39}}{2}t\right) + (1.31\times10^{-6})\sin sin\left(\frac{\sqrt{39}}{2}t\right)\right) - (0.00000405)cos(20\pi t)$$

$$- (0.000254)sin(20\pi t)$$

When the spring constant, k = 100N/m, the particular solution is

$$x = e^{-\frac{1}{2}t}\left((0.00000424)\cos cos\left(\frac{\sqrt{399}}{2}t\right) + (2.57\times10^{-6})\sin sin\left(\frac{\sqrt{399}}{2}t\right)\right) - (0.00000424)cos(20\pi t)$$

$$- (0.000254)sin(20\pi t)$$



Red: 1 N/m Black: 10 N/m Green: 100 N/m

It seems that the increase in spring constant does not affect the variation of displacement of a building and period. However, when the *k* value is made to 20π2, the =-0.0159, μ=0, A=0.0159 and B=-0.4102. At this value of *k*, amplitude of oscillation dramatically increases, which can be explained by a physical phenomenon called "resonance" which is the increase in amplitude when applied frequency is close to natural frequency(Billah).

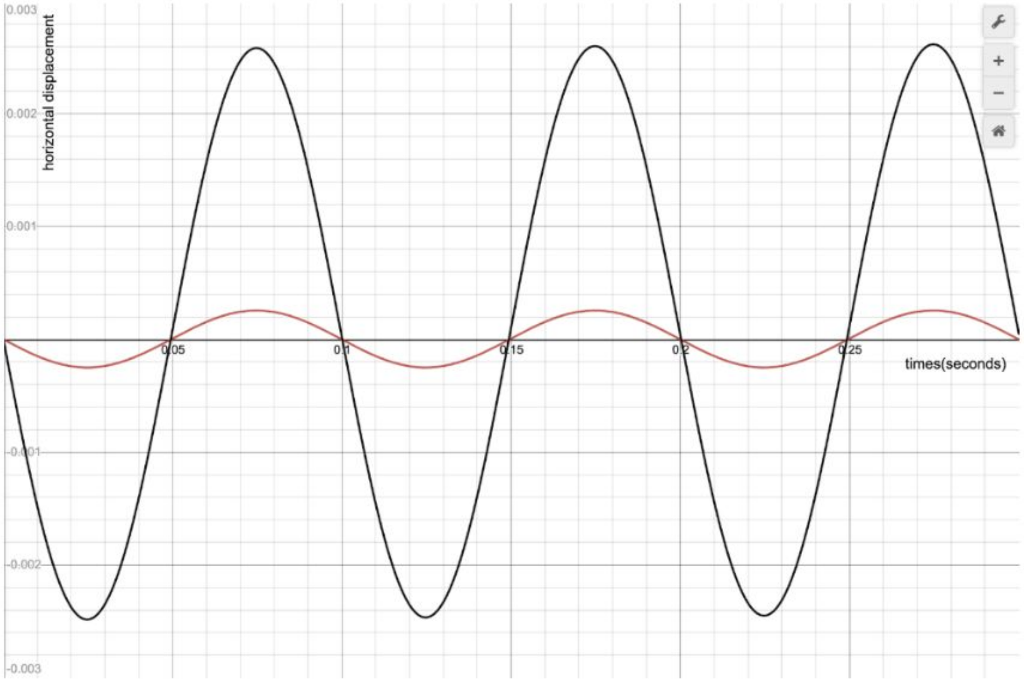
Variation of External Force

When the external force applied, F, is 10N and other variables are all 1, the general solution is

$$x = e^{-\frac{1}{2}t}\left((4.03\times10^{-5})\cos cos\left(\frac{\sqrt{3}}{2}t\right) + (0.00026)\sin sin\left(\frac{\sqrt{3}}{2}t\right)\right) - (4.03\times10^{-5})cos(20\pi t) - (0.00253)sin(20\pi t)$$

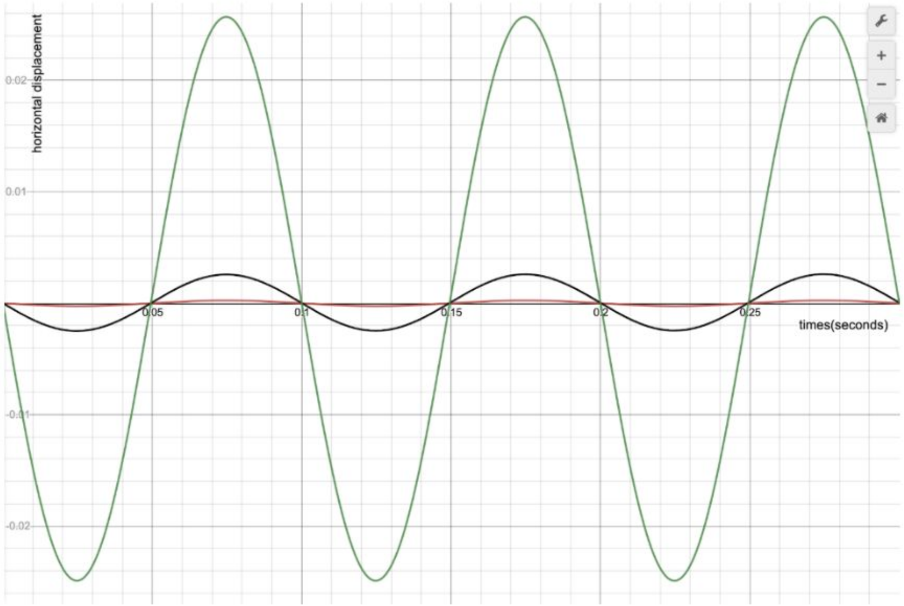
When the external force is 100N, the particular solution is

$$x = e^{-\frac{1}{2}t}\left((4.03\times10^{-4})\cos cos\left(\frac{\sqrt{3}}{2}t\right) + (0.000256)\sin sin\left(\frac{\sqrt{3}}{2}t\right)\right) - (4.03\times10^{-4})cos(20\pi t) - (0.0253)sin(20\pi t)$$



Red: 1 N Black: 10 N

Graph showing the displacement of the mass when external force is set as 1 N and 10 N



Green: 100 N/m

When the external force on the building due to the earthquake increases, the horizontal displacement also increases drastically. This indicates that the magnitude of force largely affects the amplitude of oscillation, which eventually can lead to increased chance of collapse of buildings.

Variation of damping coefficient

When the damping coefficient is varied alone, the complementary function changes the form. This is because if c is increased while others remain as 1, the discriminant of the auxiliary equation is greater than 0 and thus there would be two distinct real roots, rather than a set of complex roots, for the auxiliary equation.

The root α becomes

$$\alpha = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

This gives us the new complimentary form,

$$x = Ae^{\left(\frac{-c}{2m} + \frac{\sqrt{c^2-4mk}}{2m}\right)t} + Be^{\left(\frac{-c}{2m} - \frac{\sqrt{c^2-4mk}}{2m}\right)t}$$

By substituting $\omega = 20\pi,\ F = 1,\ m = 1,\ k = 1,\ and\ c = 10,$

$$x = Ae^{\left(-5+\frac{\sqrt{96}}{2}\right)t} + Be^{\left(-5-\frac{\sqrt{96}}{2}\right)t} \qquad (eq7)$$

The particular integral will have the same form as ***eq*2**. So substituting the same values as before, ***eq*3** and ***eq*4** becomes

$$\left(1-(20\pi)^2\right)\lambda + 10\times(20\pi)\mu = 0 \qquad (eq8)$$
$$\left(1-(20\pi)^2\right)\mu - 10\times(20\pi)\lambda = 1 \qquad (eq9)$$

Solving this simultaneous equation, I get $\lambda =- 3.933\times10^{-5}$, $\mu =- 0.000247$, and so the particular integral becomes

$$x =- 3.933\times10^{-5}cos(20\pi t) - 0.000247sin(20\pi t)$$

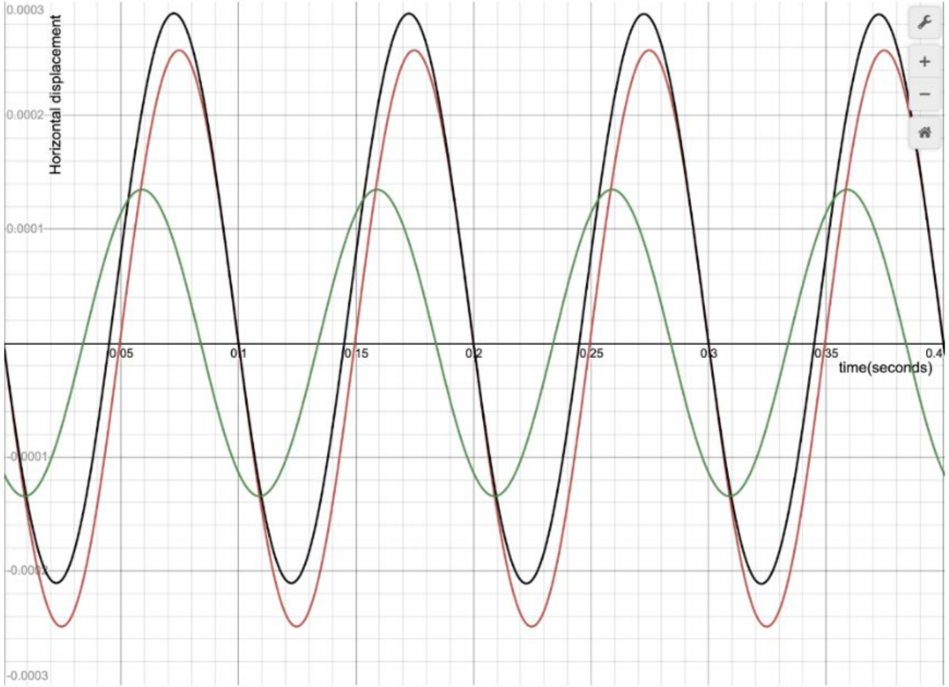
Using the same boundary conditions, (when $t = 0,\ x = 0$ and when $t = \frac{1}{10}, x = 0$)

$A = 3.996\times10^{-5}$ and $B =- 6.39\times10^{-7}$

$$x = (3.996\times10^{-5})e^{\left(-5+\frac{\sqrt{96}}{2}\right)t} + (-6.39\times10^{-7})e^{\left(-5-\frac{\sqrt{96}}{2}\right)t} - 3.933\times10^{-5}cos(20\pi t) - 0.000247sin(20\pi t)$$

Similarly, when $\omega = 20\pi,\ F = 1,\ m = 1,\ k = 1,\ and\ c = 100,$

$$x = (1.14\times10^{-7})e^{\left(-50+\frac{\sqrt{9996}}{2}\right)t} + (-1.14\times10^{-4})e^{\left(-50-\frac{\sqrt{9996}}{2}\right)t} - (1.14\times10^{-4})cos(20\pi t) - (7.168\times10^{-5})sin(20\pi t)$$



Red: 1 N s/m Black: 10 N s/m Green: 100 N s/m

As the damping coefficient increases from 1 to 10 N s/m, the amplitude of the oscillation increases. This can also be explained by resonance, hinting that at some specific value of *c*, the system's natural frequency is approximately equal to the driving frequency of 10 Hz, causing the amplitude of the oscillation to increase dramatically. As the damping coefficient is increased further from 10 N s/m to 100 N s/m, the amplitude decreases, supporting the idea of resonance.

Solving again through Laplace Inverse Transform(matlab)

Reasons for Using the Inverse Laplace Transform

- **Simplifies Complex Calculation**s: The inverse Laplace transform makes it easier to solve second-order differential equations by converting them from the time domain to the s-domain, where they become algebraic equations.
- **Enables Analytical Solutions:** Allows for the derivation of exact solutions to differential equations, facilitating a deeper understanding of the system dynamics.
- **Verification Tool**: It is used to verify solutions obtained manually, ensuring accuracy and consistency in the mathematical modeling of physical systems.

Finding the Inverse Laplace Transform for a Damped System

- **Differential Equation Setup**
  - Original form: $\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F\sin(\omega t)$
  - Assume initial conditions, typically $x(0) = 0$ and $x'(0) = 0$.
- **Laplace Transform Application**
  - Apply Laplace transforms to each term:
    - $\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} = s^2X(s) - sx(0) - x'(0)$
    - $\mathcal{L}\left\{\frac{dx}{dt}\right\} = sX(s) - x(0)$
    - $\mathcal{L}\{x(t)\} = X(s)$
    - $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$
- **Formulate and Simplify Laplace Equation**
  - Combine and simplify to form:
    - $(s^2 + cs + k)X(s) = \frac{\omega F}{s^2+\omega^2}$
- **Solve for $X(s)$**
  - Rearrange to isolate $X(s)$:
    - $X(s) = \frac{\omega F}{(s^2+cs+k)(s^2+\omega^2)}$

Base solution where all the variables equal to 1

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F\sin(\omega t)$$

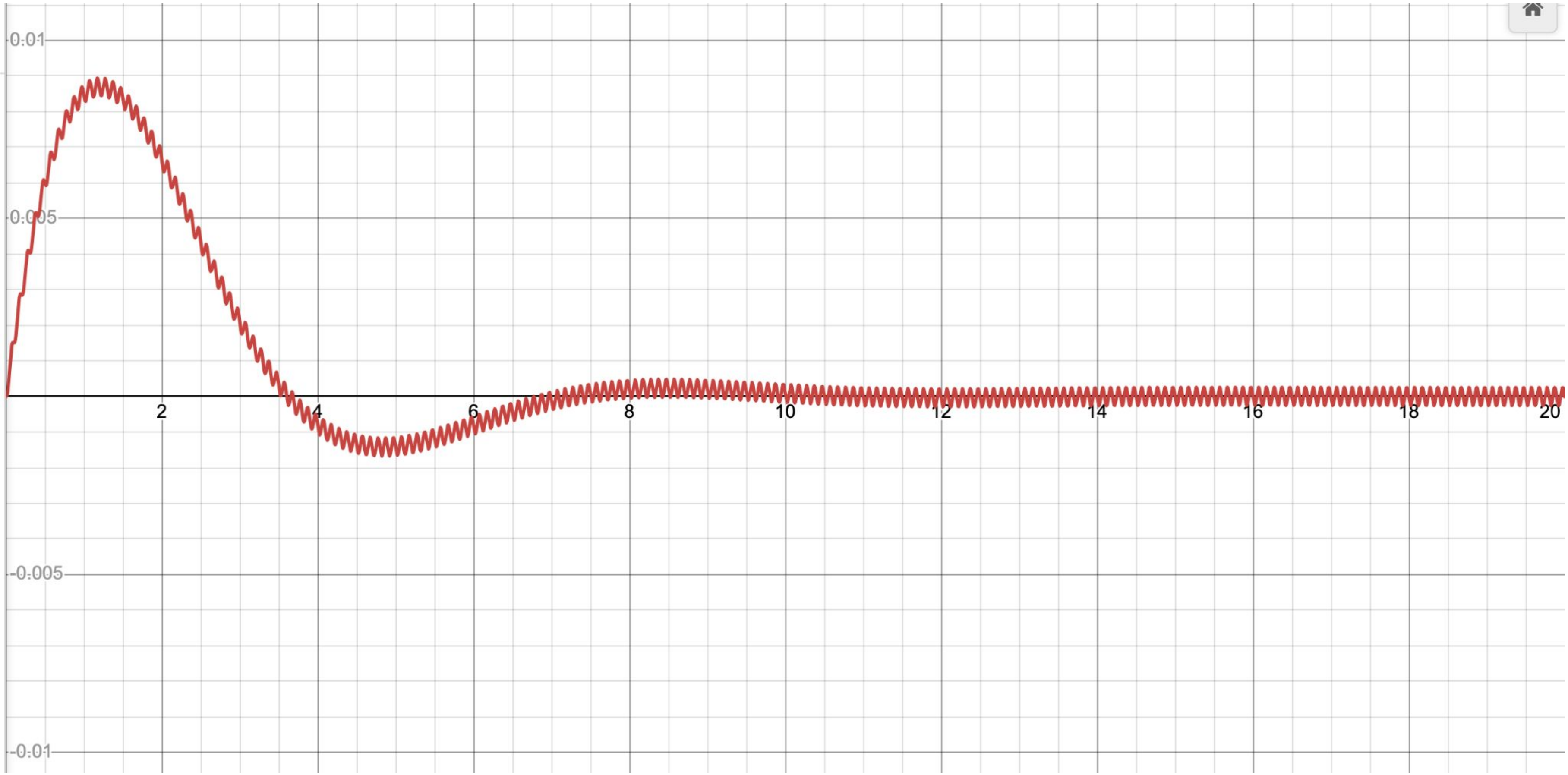
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = \sin(20\pi t)$$

$$s^2X(s) + sX(s) + X(s) = \frac{20\pi}{s^2+400\pi^2}$$

$$X(s) = \frac{20\pi}{(s^2+s+1)(s^2+400\pi^2)}$$

```
>> syms s
>> F = (20*pi)/((s^2+s+1)*(s^2+400*pi^2));
>> f = ilaplace(F)
```

$$f = \frac{9671406556917033397649408 0\cdot\pi\cdot e^{-\frac{t}{2}}\cdot\left(\cos\left(\frac{\sqrt{3}\cdot t}{2}\right)+\frac{2893432109603257\cdot\sqrt{3}\cdot\sin\left(\frac{\sqrt{3}\cdot t}{2}\right)}{1099511627776}\right)}{7534754798272584690418502919 0969} - \frac{96714065569170333976494080\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{75347547982725846904185029190969} - \frac{40025730612977692238944794029588 4800\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{654121889644404652751666537144142289136380913043}$$


0.01
0.005
-0.005
-0.01
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18
20

Variation of Mass

$$10\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = \sin(20\pi t)$$

$$10(s^2X(s)) + sX(s) + X(s) = \frac{20\pi}{s^2+400\pi^2}$$

$$X(s) = \frac{20\pi}{(10s^2+s+1)(s^2+400\pi^2)}$$

```
>> syms s
>> F = (20*pi)/((10*s^2+s+1)*(s^2+400*pi^2));
>> f = ilaplace(F)
```

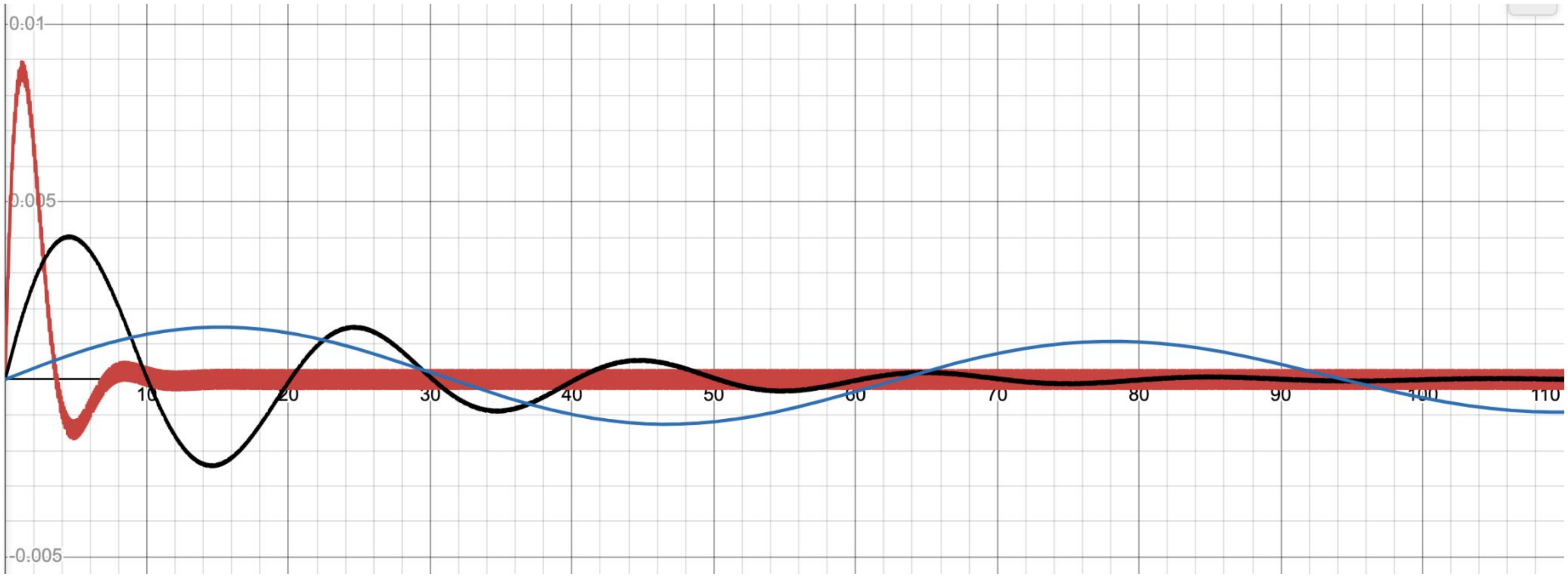
$$f = \frac{2417851639229258349412352 0\cdot\pi\cdot e^{-\frac{t}{20}}\cdot\left(\cos\left(\frac{\sqrt{39}\cdot t}{20}\right)+\frac{72343224443568913\cdot\sqrt{39}\cdot\sin\left(\frac{\sqrt{39}\cdot t}{20}\right)}{3573412790272}\right)}{188407516435920146571241199333 4417} - \frac{24178516392292583494123520\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{1884075164359201465712411993334417} - \frac{1000871442432483387265889257316679680\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{16356402294939659306097511639017584203072774155099}$$

When mass=100 kg and other variables are all 1

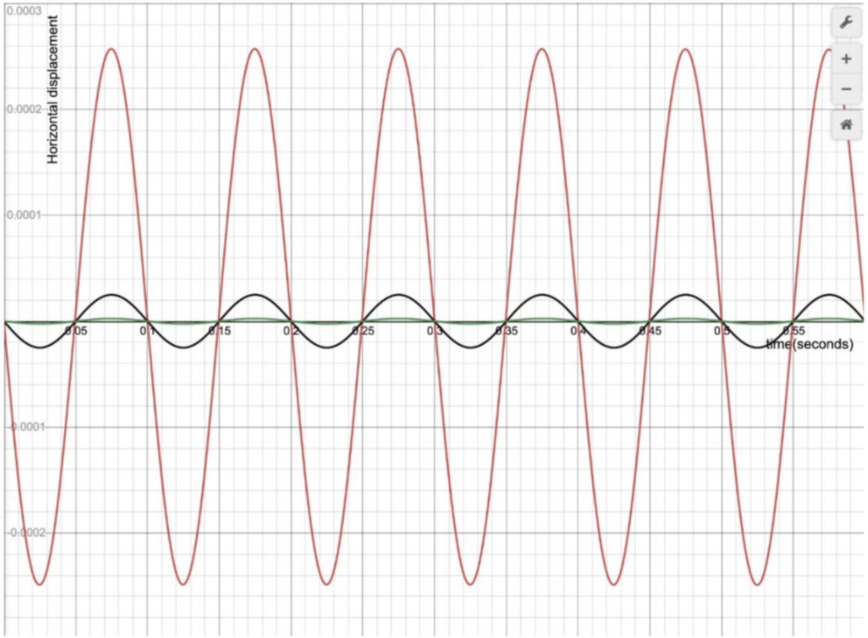
$$100s^2X(s) + sX(s) + X(s) = \frac{20\pi}{s^2+400\pi^2}$$

$$X(s) = \frac{20\pi}{(100s^2+s+1)(s^2+400\pi^2)}$$

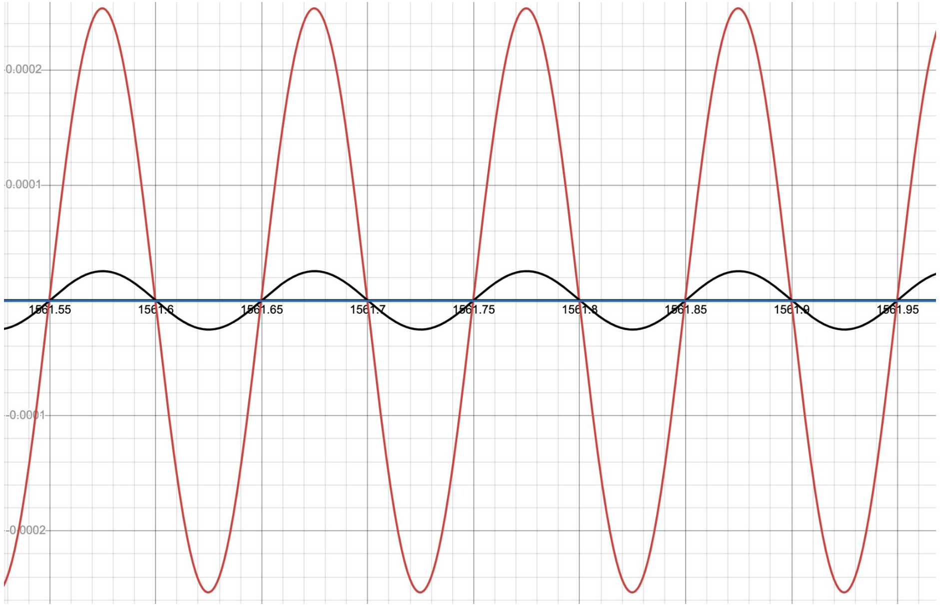
$$f = \frac{604462909807314587353088 0\cdot\pi\cdot e^{-\frac{t}{200}}\cdot\left(\cos\left(\frac{\sqrt{399}\cdot t}{200}\right)+\frac{1808619575032532137\cdot\sqrt{399}\cdot\sin\left(\frac{\sqrt{399}\cdot t}{200}\right)}{9139690405888}\right)}{47103908647551357640867619196608553} - \frac{6044629098073145873530880\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{47103908647551357640867619196608553} - \frac{25022356503582187384877904949359411 20\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{408927676601202556326790517165830338299238002539491}$$



Red: 1 kg Black: 10 kg Green: 100 kg



VS



Solving 2nd order differential equation manually

Via inverse laplace transform + matlab

Variation of spring constant

Now, when the spring constant, $k = 10\text{N/m}$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 10x = \sin(20\pi t)$$

$$X(s) = \frac{20\pi}{(s^2+s+10)(s^2+400\pi^2)}$$

```
>> syms s
>> F = (20*pi)/((s^2+s+10)*(s^2+400*pi^2));
>> f = ilaplace(F)
```

$$f = \frac{9671406556917033397649408 0\cdot\pi\cdot e^{-\frac{t}{2}}\cdot\left(\cos\left(\frac{\sqrt{39}\cdot t}{2}\right) + \frac{2886835039836601\cdot\sqrt{39}\cdot\sin\left(\frac{\sqrt{39}\cdot t}{2}\right)}{14293651161088}\right)}{75004396073162472114160826639673} -$$

$$\frac{96714065569170333976494080\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{75004396073162472114160826639673} -$$

$$\frac{39934459769761259722037031398801408 0\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{6511428520840829695446158256667530883740290021 31}$$

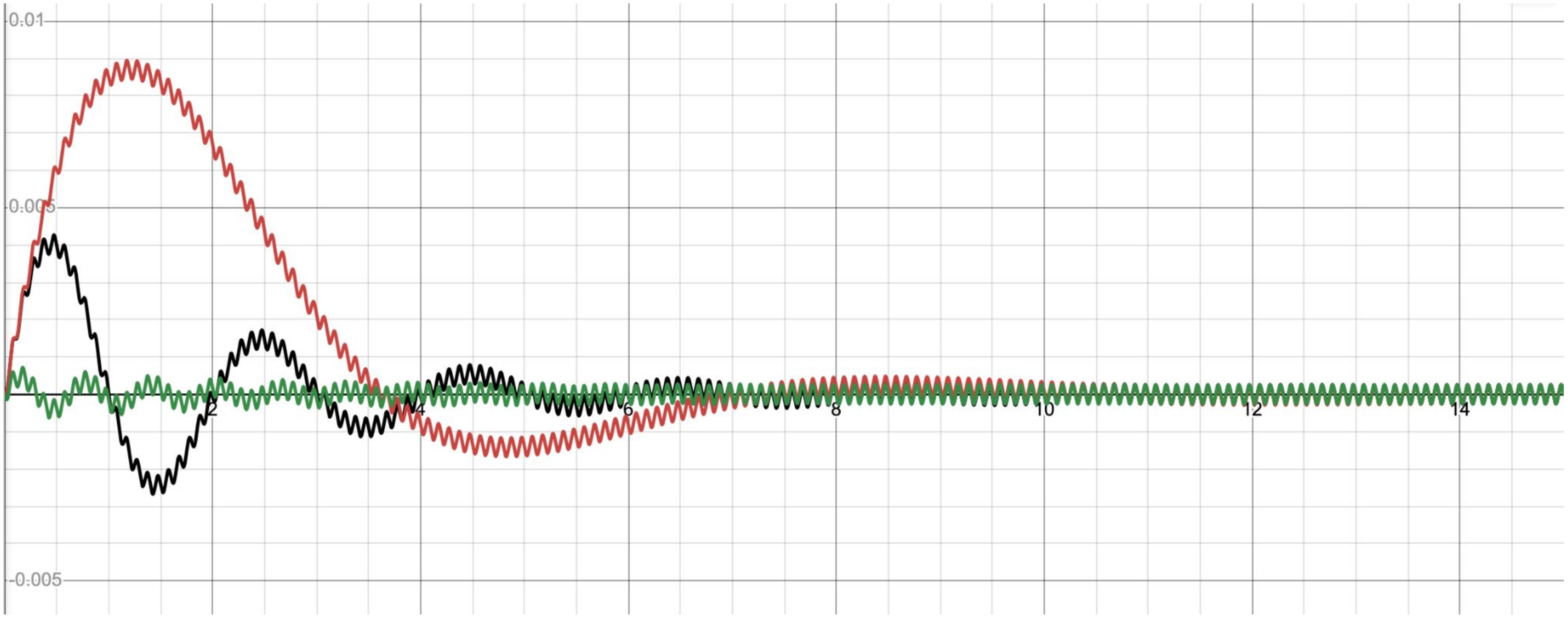
When the spring constant, k = 100N/m

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 100x = \sin(20\pi t)$$

$$X(s) = \frac{20\pi}{(s^2+s+100)(s^2+400\pi^2)}$$

```
>> syms s
>> F = (20*pi)/((s^2+s+100)*(s^2+400*pi^2));
>> f = ilaplace(F)
```

$$f = \frac{9671406556917033397649408 0\cdot\pi\cdot e^{-\frac{t}{2}}\cdot\left(\cos\left(\frac{\sqrt{399}\cdot t}{2}\right) + \frac{148466544324739\cdot\sqrt{399}\cdot\sin\left(\frac{\sqrt{399}\cdot t}{2}\right)}{7696581394432}\right)}{7161596309373978959770532923935 3} - \frac{96714065569170333976494080\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{71615963093739789597705329239353} - \frac{390217513375969345529594050909306880\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{6217265241109214892597708825234566721842111 87091}$$



0.01
0.005
-0.005
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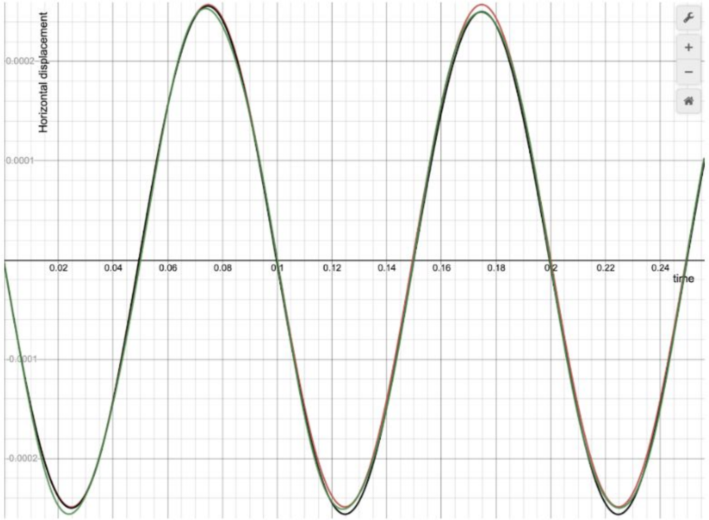
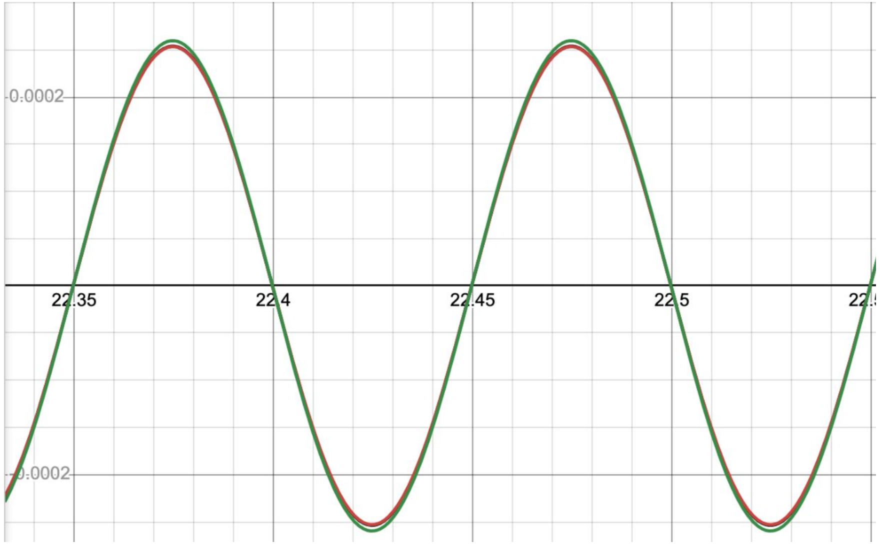
14



VS



Solving 2nd order differential equation manually

Via inverse laplace transform + matlab

Variation of External Force

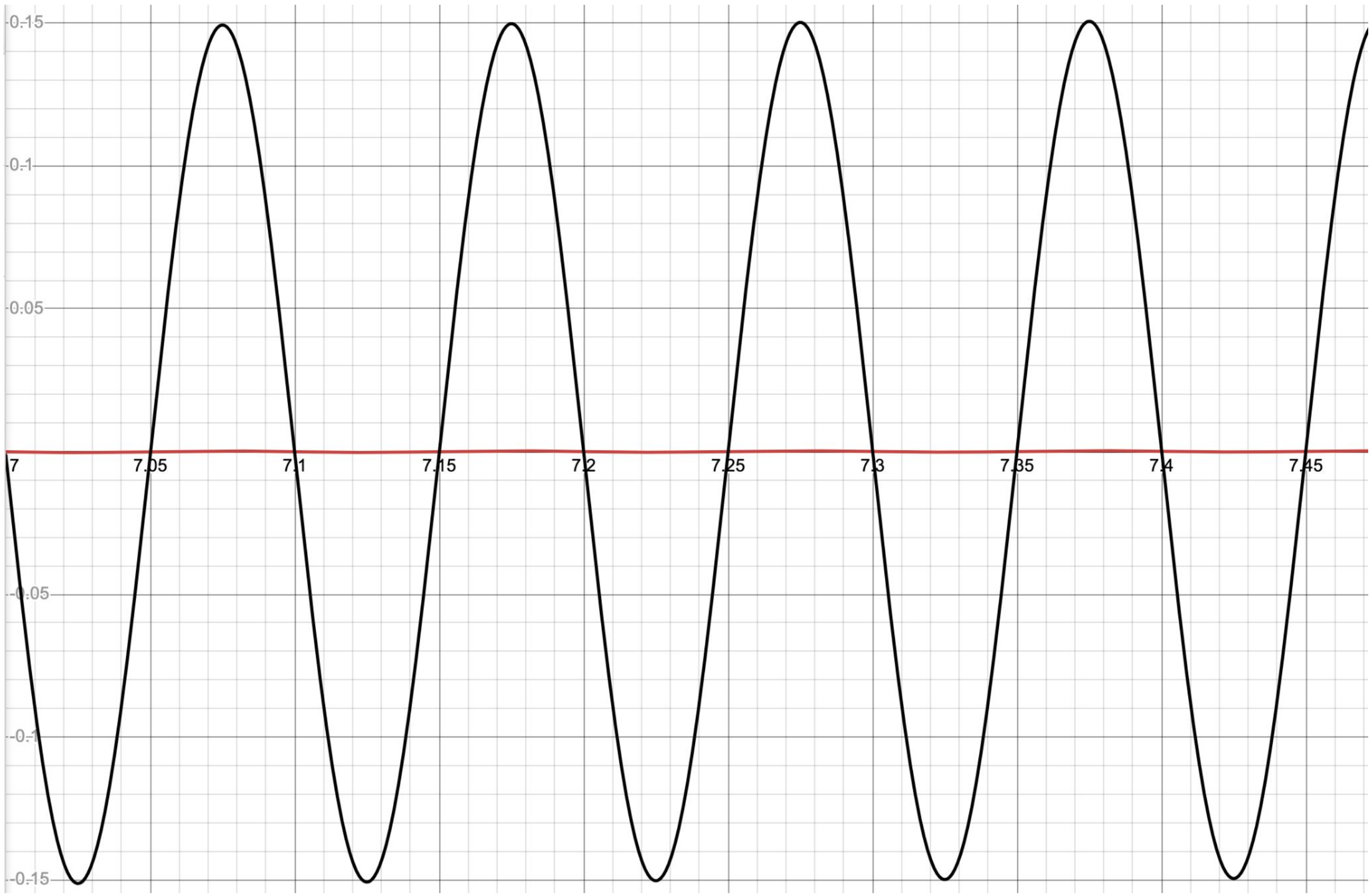
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 10\sin(20\pi t)$$

$$X(s) = \frac{200\pi}{(s^2+s+1)(s^2+400\pi^2)}$$

$$f = \frac{967140655691703339764940800\cdot\pi\cdot e^{-\frac{t}{2}}\cdot\left(\cos\left(\frac{\sqrt{3}\cdot t}{2}\right)+\frac{2893432109603257\cdot\sqrt{3}\cdot\sin\left(\frac{\sqrt{3}\cdot t}{2}\right)}{1099511627776}\right)}{75347547982725846904185029190969} -$$

$$\frac{967140655691703339764940800\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{75347547982725846904185029190969} -$$

$$\frac{4002573061297769223894479402958848000\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{654121889644404652751666537144142289136380913043}$$


0.15
0.1
0.05
-0.05
-0.1
-0.15
7
7.05
7.1
7.15
7.2
7.25
7.3
7.35
7.4
7.45

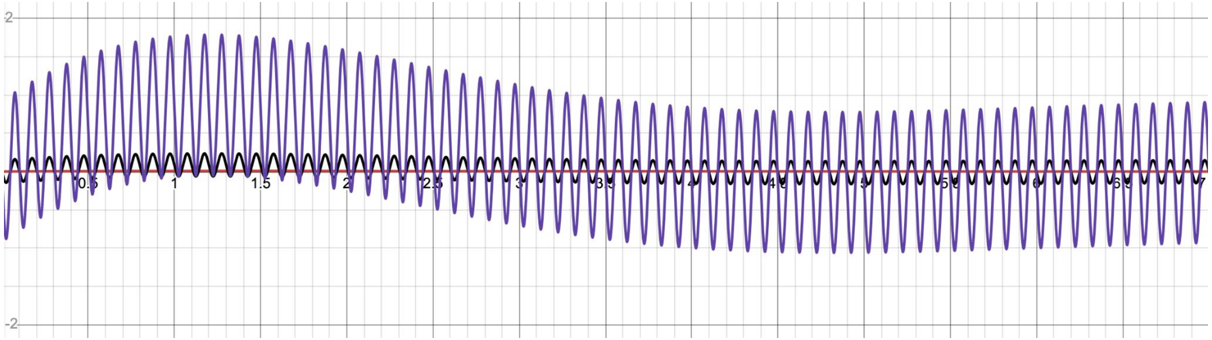
Now when F = 100

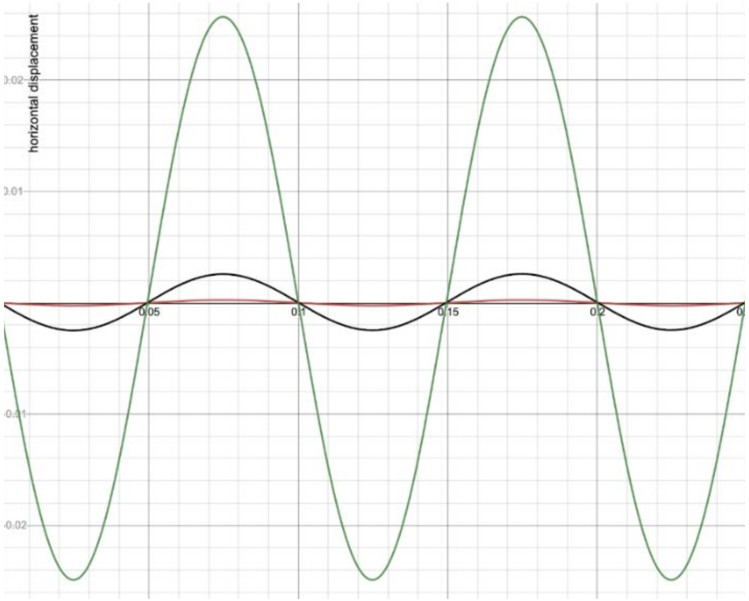
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 100\sin(20\pi t)$$

$$X(s) = \frac{2000\pi}{(s^2+s+1)(s^2+400\pi^2)}$$

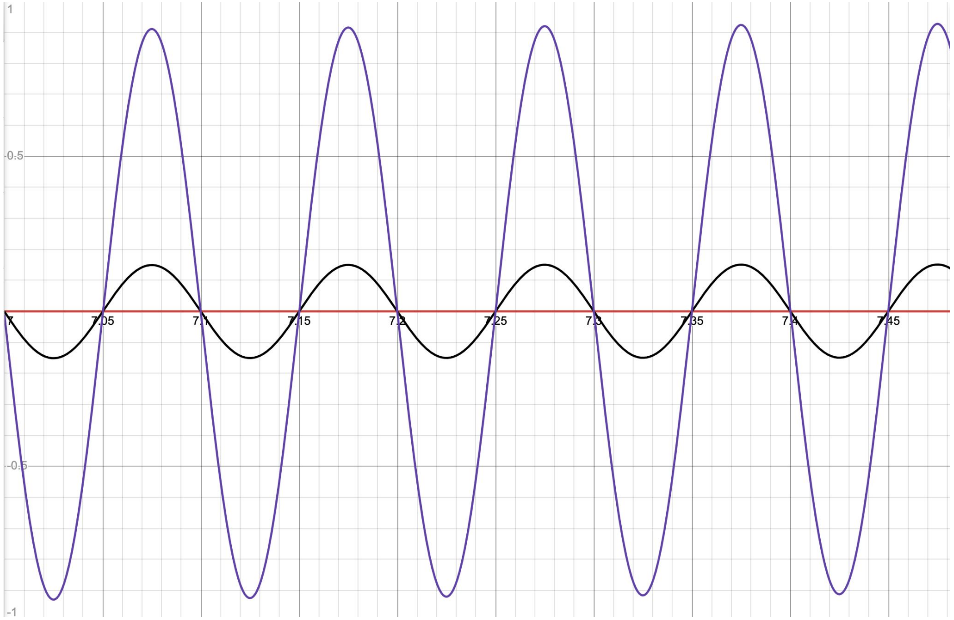
```
>> syms s
>> F = (2000*pi)/((s^2+s+1)*(s^2+400*pi^2));
>> f = ilaplace(F)
```

$$f = \frac{967140655691703339764940800 0\cdot\pi\cdot e^{-\frac{t}{2}}\cdot\left(\cos\left(\frac{\sqrt{3}\cdot t}{2}\right)+\frac{2893432109603257\cdot\sqrt{3}\cdot\sin\left(\frac{\sqrt{3}\cdot t}{2}\right)}{1099511627776}\right)}{75347547982725846904185029190969} - \frac{9671406556917033397649408000\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{75347547982725846904185029190969} - \frac{40025730612977692238944794029588480000\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{6541218896444046527516665371441422891363809130 43}$$


2
-2
0.5
1
1.5
2
2.5
7


horizontal displacement
0.02
0.01
-0.01
-0.02
0.05
0.1
0.15
0.2

VS


1
0.5
-0.5
-1
7
7.05
7.1
7.15
7.2
7.25
7.3
7.35
7.4
7.45

Variation of damping coefficient

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + x = \sin(20\pi t)$$

$$X(s) = \frac{20\pi}{(s^2+10s+1)(s^2+400\pi^2)}$$

```
>> syms s
>> F = (20*pi)/((s^2+10*s+1)*(s^2+400*pi^2));
>> f = ilaplace(F)
```

$$f = \frac{386856262276681335905976 32\cdot\pi\cdot e^{-5t}\cdot\left(\cosh(2\cdot\sqrt{6}\cdot t)+\frac{585943198663973\cdot\sqrt{6}\cdot\sinh(2\cdot\sqrt{6}\cdot t)}{17592186044416}\right)}{3089500661030381156466426130977} - \frac{3868562622766813359059763 2\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{3089500661030381156466426130977} - \frac{160102922451910768955779176118353 92\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{26821178187698202831093881096999438290010593419}$$

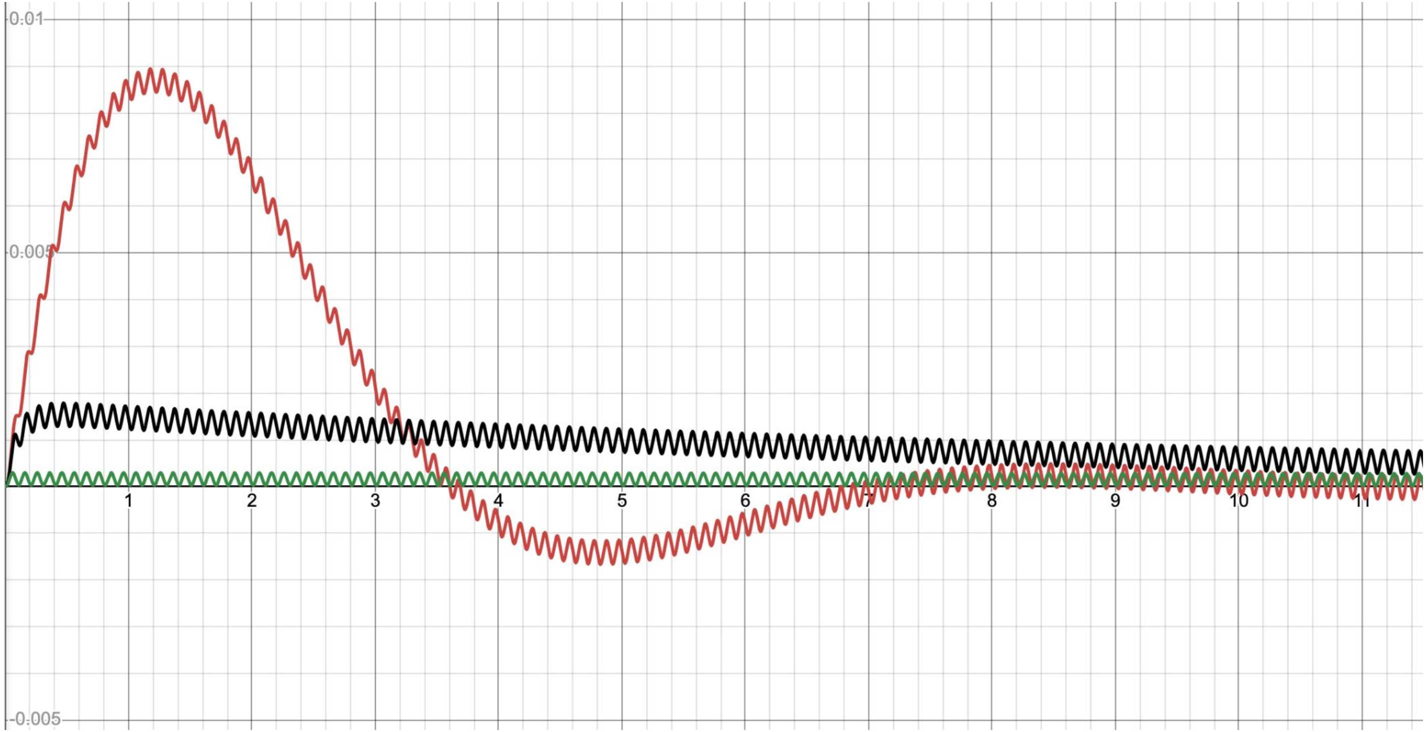
When damping coefficient = 100

$$\frac{d^2x}{dt^2} + 100\frac{dx}{dt} + x = \sin(20\pi t)$$

$$X(s) = \frac{20\pi}{(s^2+100s+1)(s^2+400\pi^2)}$$

```
>> syms s
>> F = (20*pi)/((s^2+100*s+1)*(s^2+400*pi^2));
>> f = ilaplace(F)
```

$$f = \frac{386856262276681335905976320\cdot\pi\cdot e^{-50t}\cdot\left(\cosh(7\cdot\sqrt{51}\cdot t)+\frac{187374410428019\cdot\sqrt{51}\cdot\sinh(7\cdot\sqrt{51}\cdot t)}{747667906887680}\right)}{1064937483316510918636892246480 1} - \frac{386856262276681335905976320\cdot\pi\cdot\cos\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{10649374833165109186368922464801} - \frac{16010292245191076895577917611835392\cdot 2^{\frac{1}{2}}\cdot 8681395840437547^{\frac{1}{2}}\cdot\pi\cdot\sin\left(\frac{\sqrt{2}\cdot 8681395840437547^{\frac{1}{2}}\cdot t}{2097152}\right)}{924514383798998749338158422203741107455462831 47}$$



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-0.005
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Evaluation and Conclusion

Evaluation & Limitations

1. **Emphasis on Initial Conditions:** The Inverse Laplace transform solution distinctly highlights the initial shock wave, which can be seen in the first graph. This pronounced peak suggests a more sensitive response to initial conditions when utilizing the Inverse Laplace method, which could be attributed to the mathematical precision and inherent nature of the Laplace domain where initial conditions are explicitly accounted for in the solution.

2. **Convergence Over Time:** As time progresses, the solutions converge, which is apparent when comparing the first graph to the second. This indicates that the transient effects, which are initially more pronounced in the Inverse Laplace transform solution, diminish over time, leading to a steady-state behavior that is consistent with the manual solution.

3. **Time-Invariance:** The Laplace transform assumes time-invariant systems, where the parameters (mass, spring constant, damping coefficient) do not change over time. In actual situations, these properties can vary due to external factors such as temperature, wear and tear, or damage.

4. **Initial Conditions:** The analysis assumes that initial conditions are known and can be exactly defined. In reality, measuring initial conditions with absolute precision is challenging, and slight variations can lead to significantly different outcomes, particularly when examining the system's transient response.

Conclusion

The investigation into the dynamic response of a building during seismic events has yielded insights with substantial real-world implications. By leveraging the analytical prowess of the Laplace transform and manual solving techniques, we've confirmed that while mathematical models are robust for long-term behavior predictions, the nuanced transient responses—especially initial shock waves—require sophisticated computational approaches like MATLAB for precision. This distinction is particularly critical in engineering where early-time responses can dictate the resilience of a structure during an earthquake. Furthermore, the study has underlined the significance of selecting system parameters like spring constants and damping coefficients to mitigate the risks associated with resonance—a phenomenon that can amplify forces to catastrophic levels. Therefore, this research contributes to the field of seismic design, providing a foundation for engineering safer, more reliable structures that can withstand the unpredictability and force of seismic waves, ultimately safeguarding human lives and infrastructure.